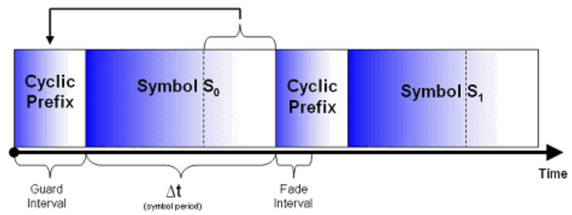

ECS455: Chapter 5

OFDM

5.4 Cyclic Prefix (CP)

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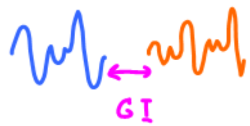
Three steps towards modern OFDM

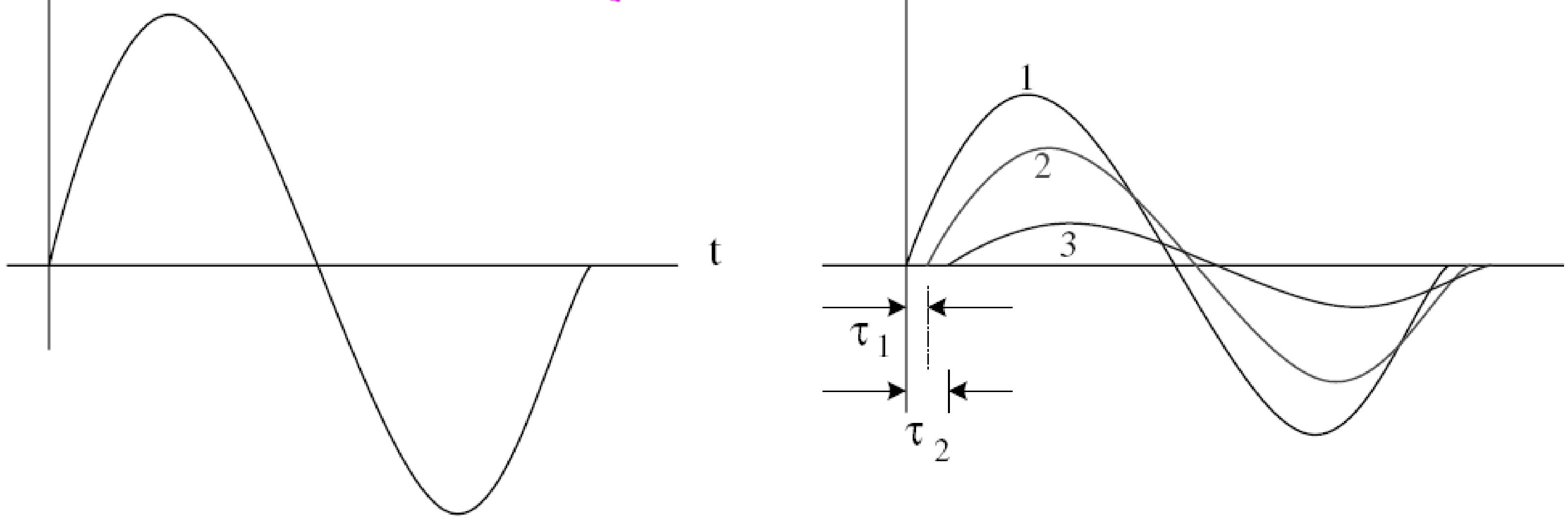
1. Mitigate Multipath (ISI): Decrease the rate of the original data stream via multicarrier modulation (FDM)
2. Gain Spectral Efficiency: Utilize orthogonality
3. Achieve Efficient Implementation: FFT and IFFT

- Extra step: Completely eliminate ISI and ICI
  - Cyclic prefix

Cyclic Prefix: Motivation (1)

- Recall: Multipath Fading and Delay Spread





Transmitted Signal

Received Signal

Cyclic Prefix: Motivation (2)

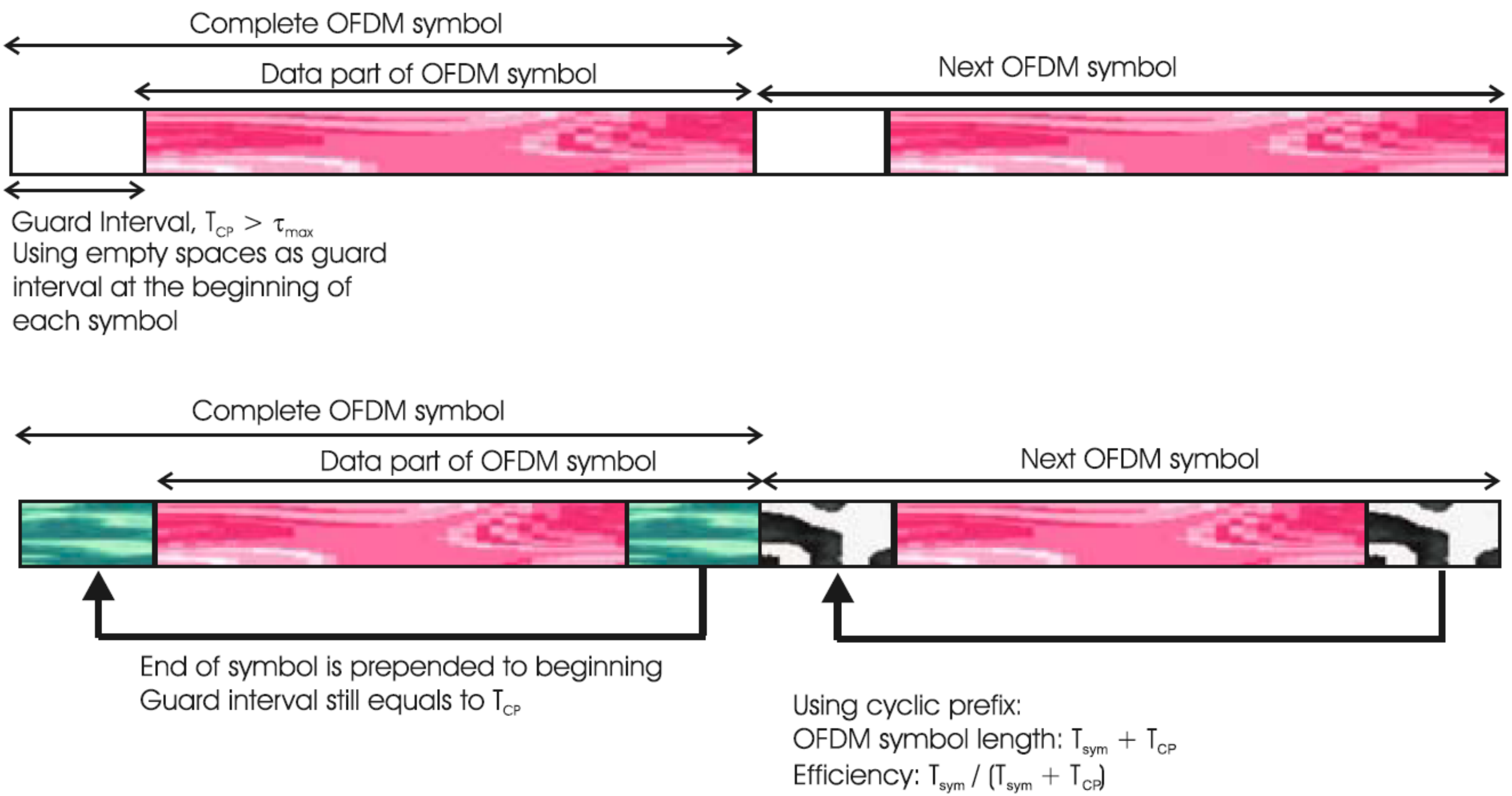
- OFDM uses large symbol duration T_s
  - compared to the duration of the impulse response $\tau_{\max}$ of the channel
  - to reduce the amount of ISI
- **Q**: Can we "eliminate" the multipath (**ISI**) problem?
- **A**: To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- **Solution**: To prevent **both** the **ISI** as well as the **ICI**, OFDM symbol is **cyclically extended** into the guard interval.

$c_1(t) \perp c_2(t)$

$\not\Downarrow$

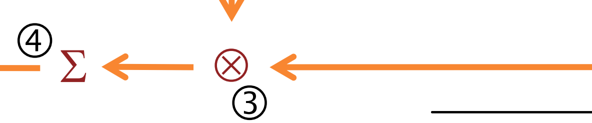
$c_1(t-\tau_1) \perp c_2(t-\tau_2)$

Cyclic Prefix

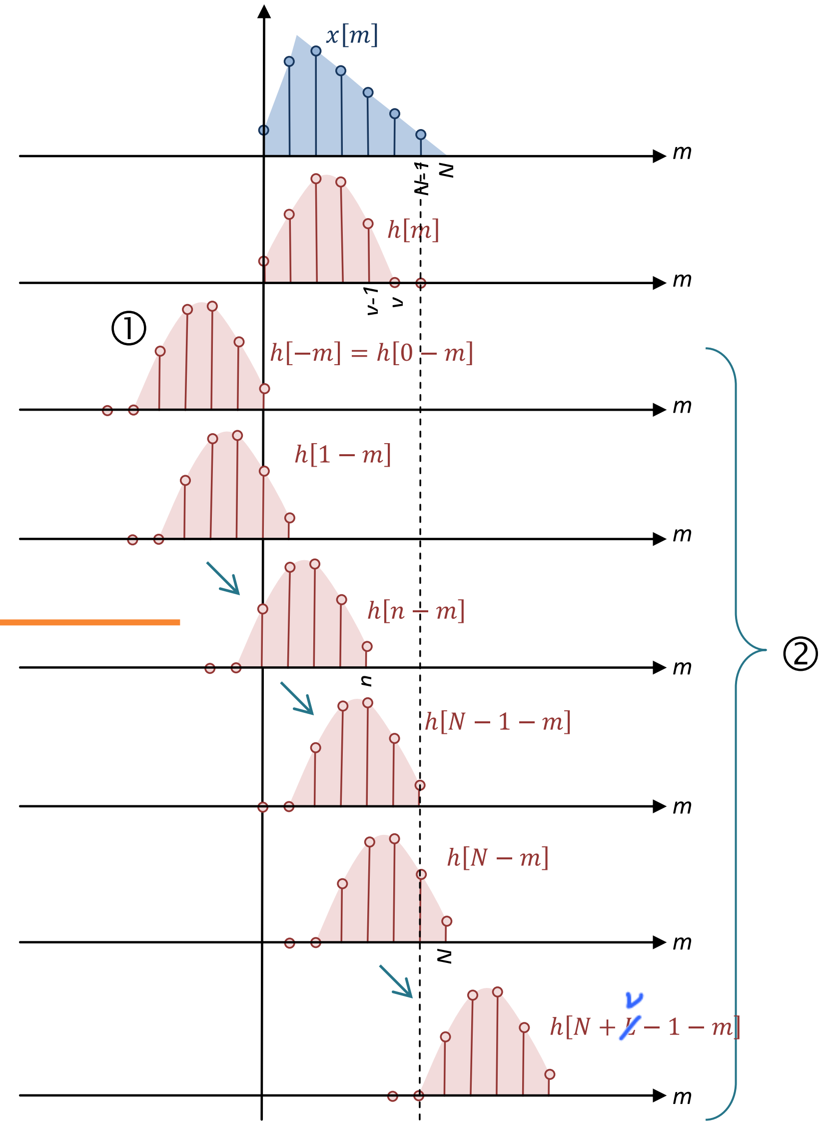


Recall: Convolution

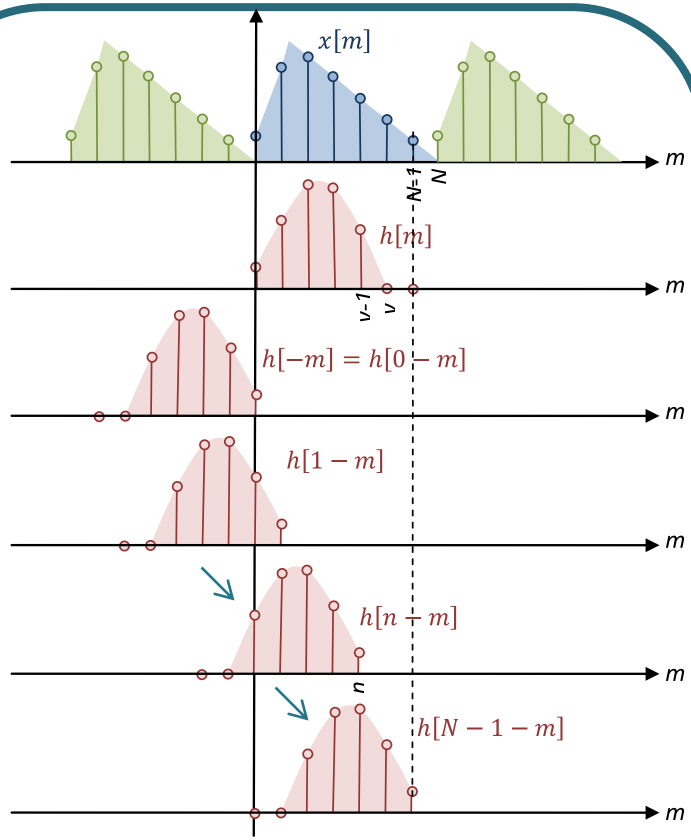
① Flip
② Shift
③ Multiply (pointwise)
④ Add



$$\{x * h\}[n] = \sum_m x[m]h[n-m]$$

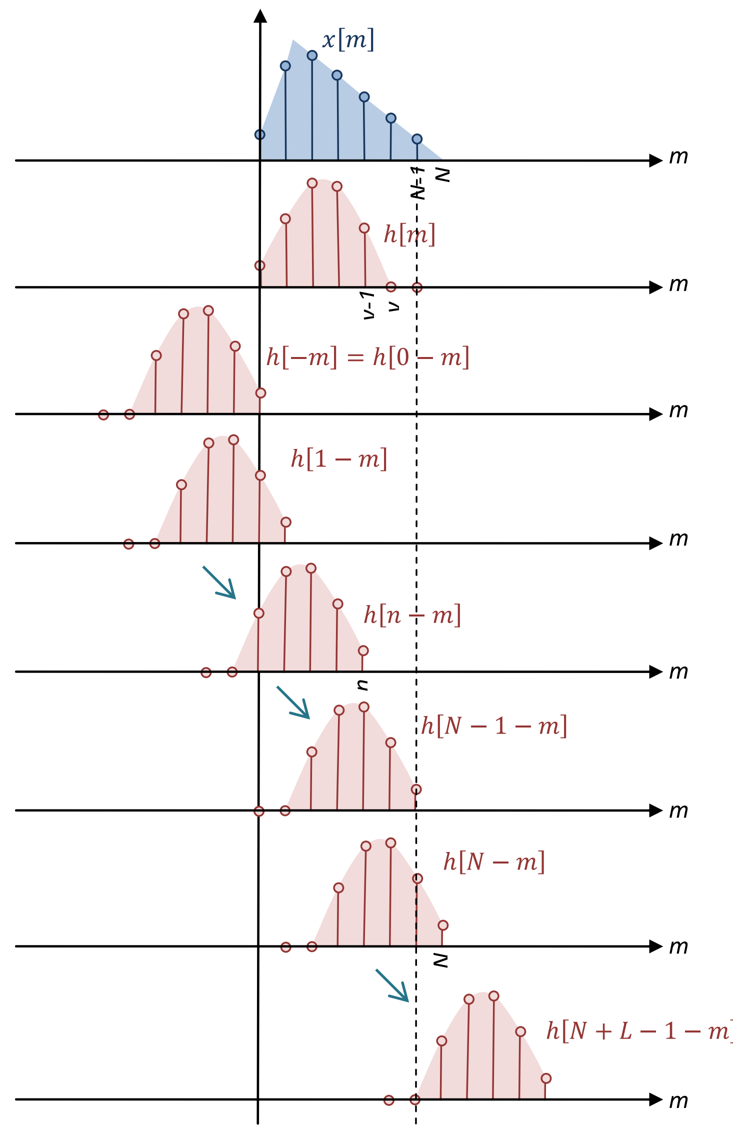


Circular Convolution



Replicate x (now it looks periodic)
Then, perform the usual convolution
only on n = 0 to N-1

(Regular Convolution)



Circular Convolution: Examples 1

Find

regular convolution

$$[1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

$= [4 \quad 13 \quad 28 \quad 27 \quad 18]$

1 2 3
6 5 4
6 5 4
6 5 4
6 5 4
6 5 4

4
5+8 = 13
6+10+12 = 28
12+15 = 27
18

$$[1 \quad 2 \quad 3] \circledast [4 \quad 5 \quad 6]$$

$= [31 \quad 31 \quad 28]$

1 2 3 1 2 3 1 2 3 1 2 3 1 2 3
6 5 4
6 5 4
6 5 4

12+15+4 = 31
18+5+8 = 31
6+10+12 = 28

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0] = [4 \quad 13 \quad 28 \quad 27 \quad 18]$$

Discussion

- ***Regular convolution*** of an N_1–point vector and an N_2–point vector gives $(N_1+N_2\text{-}1)$-point vector.
- ***Circular convolution*** is performed between two equal-length vectors. The results also has the same length.
- Circular convolution can be used to find the regular convolution by **zero-padding**.
  - Zero-pad the vectors so that their length is $N_1+N_2\text{-}1$.
  - Example:

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0] = [1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

- In modern OFDM, we want to perform circular convolution via regular convolution.

Circular Convolution in Communication

- We want the receiver to obtain the circular convolution of the signal (channel input) and the channel.
- Q: Why?
- A:
  - **CTFT**: **convolution** in time domain corresponds to **multiplication** in frequency domain.
    - This fact does not hold for DFT.
  - **DFT**: circular **convolution** in (discrete) time domain corresponds to **multiplication** in (discrete) frequency domain.
    - We want to have multiplication in frequency domain.
    - So, we want circular convolution and not the regular convolution.
- Problem: Real channel does regular convolution.
- Solution: With **cyclic prefix**, regular convolution can be used to create circular convolution.

Example 2

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Let's look closer at how we carry out the circular convolution operation. Recall that we replicate the x and then perform the regular convolution (for N points)

Solution:

$$\begin{array}{ccccccccccccccc}
1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 & 1 & -2 & 3 & 1 & 2 \\
 & 0 & 0 & 1 & 2 & 3 & & & & & & & & & \\
 & & 0 & 0 & 1 & 2 & 3 & & & & & & & & \\
 & & & 0 & 0 & 1 & 2 & 3 & & & & & & & \\
 & & & & 0 & 0 & 1 & 2 & 3 & & & & & & \\
 & & & & & 0 & 0 & 1 & 2 & 3 & & & & &
\end{array}$$

$$1\times1+2\times2+1\times3=1+4+3=8$$

$$2\times1+1\times2+(-2)\times3=2+2-6=-2$$

$$1\times1+(-2)\times2+3\times3=1-4+9=6$$

$$(-2)\times1+3\times2+1\times3=-2+6+3=7$$

$$3\times1+1\times2+2\times3=3+2+6=11$$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = \underbrace{[8 \quad -2 \quad 6 \quad 7 \quad 11]}_{\text{Goal: Get these numbers using regular convolution}}$$

Example 2

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Observation: We don't need to replicate the x indefinitely. Furthermore, when h is shorter than x, we need only a part of one replica.

Not needed in the calculation

$$\boxed{1 \quad -2 \quad 3} \; \boxed{1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2} \; \boxed{1 \quad -2 \quad 3 \quad 1 \quad 2}$$

0	0	1	2	3						$1\times1+2\times2+1\times3=1+4+3=8$
	0	0	1	2	3					$2\times1+1\times2+(-2)\times3=2+2-6=-2$
		0	0	1	2	3				$1\times1+(-2)\times2+3\times3=1-4+9=6$
			0	0	1	2	3			$(-2)\times1+3\times2+1\times3=-2+6+3=7$
				0	0	1	2	3		$3\times1+1\times2+2\times3=3+2+6=11$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = [8 \quad -2 \quad 6 \quad 7 \quad 11]$$

Example 2

Try this: use only the necessary part of the replica and then convolute (regular convolution) with the channel.

$$[1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2]*[3 \quad 2 \quad 1] = ?$$

Copy the **last ν samples** of the symbols at the **beginning** of the symbol.
This partial replica is called the **cyclic prefix**.

$$
\begin{array}{ccccccccccc}
 & & 1 & 2 & 1 & -2 & 3 & 1 & 2 & & \\
1 & 2 & 3 & & & & & & & & \\
 & 1 & 2 & 3 & & & & & & & \\
 & & 1 & 2 & 3 & & & & & & \\
 & & & 1 & 2 & 3 & & & & & \\
 & & & & 1 & 2 & 3 & & & & \\
 & & & & & 1 & 2 & 3 & & & \\
 & & & & & & 1 & 2 & 3 & & \\
 & & & & & & & 1 & 2 & 3 & \\
 & & & & & & & & 1 & 2 & 3
\end{array}
$$

$$1\times 3 = 3$$
$$1\times 2 + 2\times 3 = 2 + 6 = 8$$
$$1\times 1 + 2\times 2 + 1\times 3 = 1 + 4 + 3 = 8$$
$$2\times 1 + 1\times 2 + (-2)\times 3 = 2 + 2 - 6 = -2$$
$$1\times 1 + (-2)\times 2 + 3\times 3 = 1 - 4 + 9 = 6$$
$$(-2)\times 1 + 3\times 2 + 1\times 3 = -2 + 6 + 3 = 7$$
$$3\times 1 + 1\times 2 + 2\times 3 = 3 + 2 + 6 = 11$$
$$1\times 1 + 2\times 2 = 1 + 4 = 5$$
$$2\times 1 = 2$$

Junk!

Example 2

- We now know that

$$[\underbrace{1 \quad 2}_{\text{Cyclic Prefix}} \quad 1 \quad -2 \quad 3 \quad 1 \quad 2] * [3 \quad 2 \quad 1] = [3 \quad 8 \quad \underbrace{8 \quad -2 \quad 6 \quad 7 \quad 11}_{[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 5 \quad 2]$$

- Similarly, you may check that

$$[\underbrace{-2 \quad 1}_{\text{Cyclic Prefix}} \quad 2 \quad 1 \quad -3 \quad -2 \quad 1] * [3 \quad 2 \quad 1] = [-6 \quad -1 \quad \underbrace{6 \quad 8 \quad -5 \quad -11 \quad -4}_{[2 \quad 1 \quad -3 \quad -2 \quad 1] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 0 \quad 1]$$

Example 3

- We know, from Example 2, that

```
[ 1  2  1  -2   3   1  2] * [3  2  1] = [ 3   8  8  -2   6    7  11  5  2]
```

And that

```
[-2  1  2   1  -3  -2  1] * [3  2  1] = [-6  -1  6   8  -5  -11  -4  0  1]
```

- Check that

```
  [ 1  2  1  -2  3   1   2   0   0  0  0   0    0   0] * [3  2  1]
= [ 3  8  8  -2  6   7  11   5   2  0  0   0    0   0  0  0]
```

and

```
  [ 0  0  0   0  0   0   0  -2   1  2  1  -3   -2   1] * [3  2  1]
= [ 0  0  0   0  0   0   0  -6  -1  6  8  -5  -11  -4  0  1]
```

Example 4

- We know that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Using Example 3, we have

```
     [ 1 2 1 -2 3  1  2 -2  1 2 1 -3  -2  1] * [3 2 1]

=  (  [ 1 2 1 -2 3  1  2  0  0 0 0  0   0  0]
     +[ 0 0 0  0 0  0  0 -2  1 2 1 -3  -2  1] )  * [3 2 1]

=     [ 3 8 8 -2 6  7 11  5  2 0 0  0   0  0 0 0]
     +[ 0 0 0  0 0  0  0 -6 -1 6 8 -5 -11 -4 0 1]

=     [ 3 8 8 -2 6  7 11 -1  1 6 8 -5 -11 -4 0 1]
```

Putting results together…

- Suppose $\underline{x}^{(1)}$ = [1 -2 3 1 2] and $\underline{x}^{(2)}$ = [2 1 -3 -2 1]
- Suppose $\underline{h}$ = [3 2 1]
- At the receiver, we want to get
  - [1 -2 3 1 2] ⊛ [3 2 1 0 0] = [8 -2 6 7 11]
  - [2 1 -3 -2 1] ⊛ [3 2 1 0 0] = [6 8 -5 -11 -4]
- We transmit [$\underbrace{1\ \ 2}_{\text{Cyclic prefix}}$ 1 -2 3 1 2 $\underbrace{-2\ \ 1}_{\text{Cyclic prefix}}$ 2 1 -3 -2 1].
- At the receiver, we get

  [1 2 1 -2 3 1 2 -2 1 2 1 -3 -2 1] * [3 2 1]

  = [3 8 8 -2 6 7 11 -1 1 6 8 -5 -11 -4 0 1]

Junk! To be thrown away by the receiver.

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\hat{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -v \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \hat{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} \boxed{H_k X_k}$$

OFDM with CP for Channel w/ Memory

- We want to send N samples $S_0, S_1, \ldots, S_{N-1}$ across noisy channel with memory.
- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix

$$\hat{s} = \left[s[N-\nu], \ldots, s[N-1], s[0], \ldots, s[N-1] \right]$$

- This is inputted to the channel.
- The output is

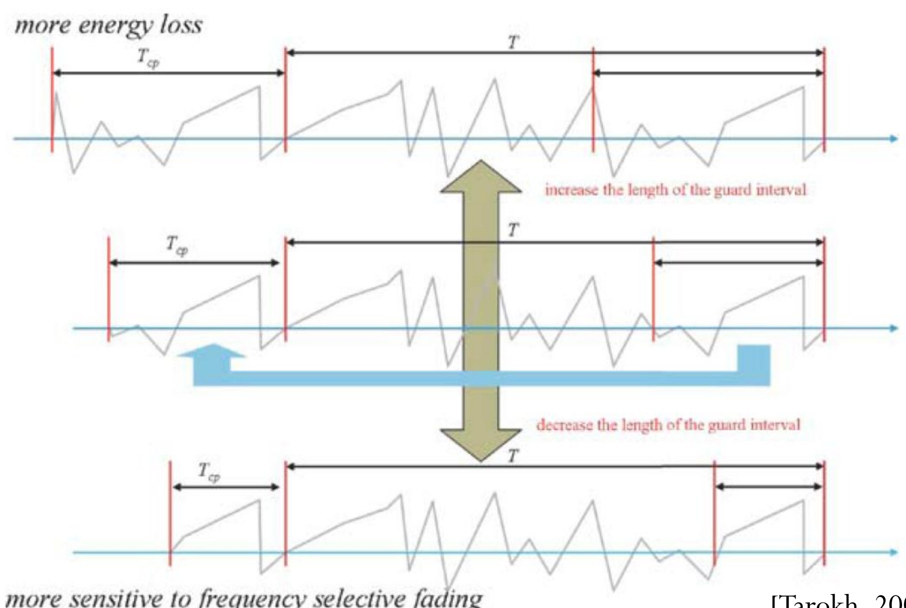
$$y[n] = \left[p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1] \right]$$

- Remove cyclic prefix to get $r[n] = h[n] \circledast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$

No ICI!

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



[Tarokh, 2009, Fig 2.9]

Summary

- The CP at the beginning of each block has two main functions.
- As guard interval, it prevents contamination of a block by ISI from the previous block.
- It makes the received block appear to be periodic of period N.
  - Turn regular convolution into circular convolution
  - Point-wise multiplication in the frequency domain

Reference

- A. Bahai, B. R. Saltzberg, and M. Ergen, ***Multi-Carrier Digital Communications: Theory and Applications of OFDM***, 2nd ed., New York: Springer Verlag, 2004.